

ANALYTICAL SIGNATURES AND PROPER ACTIONS

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ABSTRACT. In this short note we compare Mishchenko's definition of noncommutative signature for a manifold with a proper G -action of a discrete, countable group G with the (more analytical) counter part defined by Higson and Roe in the series of articles "Mapping Surgery to analysis". A generalization of the topological invariance of the coarse index is also addressed.

1. INTRODUCTION

There are different notions of non-commutative signatures that can be applied to oriented proper cocompact G -manifolds for a discrete group G . Higson and Roe studied the relation between a signature of C^* -algebras, an analytic signature and the coarse index of the signature operator, they also show that these signatures are bordism and homotopy invariants. This definitions were applied to oriented free G -Manifolds.

On the other hand, Mishchenko defined a signature associated to a proper cocompact G -manifold. In this paper we show that, with slight modifications, the C^* -algebra signature used by Higson and Roe coincides in even dimension with that of Mishchenko. The other constructions can be applied in this context to *bounded isotropy* proper G -manifolds of even dimension. A consequence of this is another proof of the homotopy and bordism invariance of the signature of Mishchenko and its coincidence with the index of the signature operator.

In [1],[2], based on the work of Mishchenko [6], Higson and Roe define signatures associated to complexes of Hilbert modules over a C^* -algebra C with values in $K_*(C)$ and it is shown that these signatures define bordism and homotopy invariants. Analogous definitions and results are given for complexes of Hilbert spaces in some *controlled* categories. These signatures have values in C^* -category K -theory of Mitchener and it is proved that this is equivalent to $K_*(C)$.

The definitions apply, in particular in the case $C = C_r^*(G)$, to the l^2 -complexes of uniform triangulations of bounded geometry G -manifolds with free action of a discrete group G . In this paper we show that, with slight modifications, these constructions can be applied to *bounded isotropy* G -manifolds with a proper action of a discrete group G .

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3. MISHCHENKO'S APPROACH TO THE SIGNATURE OF A G -MANIFOLD

In [5], a signature for the Hilbert-Poincaré complex of a proper cocompact G -manifold was defined using the Hilbert-Poincaré complex associated to an invariant triangulation taking values in $K_*(C_r^*(G))$.

Let C be a C^* -algebra. Recall that an n -dimensional Hilbert-Poincaré complex is a triple (E, d, S) where (E, d) is an n -dimensional chain complex

$$(3.1) \quad E_0 \longleftarrow E_1 \longleftarrow \cdots \longleftarrow E_{n-1} \longleftarrow E_n$$

of finitely generated Hilbert modules over a C^* -algebra C , S is a self-adjoint operator $S : E_k \rightarrow E_{n-k}$ such that $dS + Sd^* = 0$ and S induces an isomorphism from the homology of the dual complex (E, d^*) to the homology of the complex (E, d) .

Using the language and notations in [1], this definition is as follows.

Definition 3.2. (Mishchenko). Let (E, b, S) be a Hilbert-Poincaré complex of Hilbert C -modules (with S self-adjoint and $bS + Sb^* = 0$) and let $(E \oplus E, b_S)$ the mapping cone of S . Then, the signature of (E, b, S) is the formal difference $[Q_+] - [Q_-]$ in $K_0(C)$ of the positive and negative projection of the restriction of the map $B_S = b_S^* + b_S$ to $+1$ eigenspace of the isometry which exchanges the two copies of E in $E \oplus E$.

Definition 3.3. (Higson-Roe). Let (E, b, S) be an even dimensional Hilbert-Poincaré complex of Hilbert C -modules. The signature of (E, b, S) is the formal difference $[P_+] - [P_-]$ of the positive projections of $B + S$ and $B - S$ respectively.

Proposition 3.4. *Definitions 3.2 and 3.3 coincide.*

Proof. Definition (3.2) is just the index (in Kasparov approach) of the operator B_S graded by the isometry which exchanges the two copies of E in $E \oplus E$. In the proof of lemma 3.5 of [1] it is shown that B_S identifies with $B + S$ in the $+1$ eigenspace of the exchange symmetry and with $B - S$ in the -1 eigenspace. From the decomposition B_S in anti-diagonal form using this grading it is clear that the negative $B - S$ of is essentially the adjoint of $B + S$. \square

4. HIGSON-ROE'S APPROACH TO THE SIGNATURE OF A G -MANIFOLD

In this section we modify some of the notions in [1] and [2] to extend the main results there to include proper, not necessarily free actions.

Definition 4.1. Let G a finitely generated discrete group. A G -presented space X is a proper geodesic metric space presented as the quotient $X = M/G$ of a proper geodesic metric space M by an isometric proper action $\mu : G \times M \rightarrow M$ of the group G . The pair (M, μ) is called a G -presentation of X .

For a fixed discrete group G , these spaces together with equivariant morphisms of the presentations form a category. We avoid the action in the notation and say that M is a G -presentation of X . We shall assume in the following that all such presentations have an invariant open set where the action of the group G is free.

Definition 4.2. Let X be a G -presented space. An equivariant X - G -module is an M -module H , where M is a G -presentation of X , equipped with a compatible unitary representation of G .

Definition 4.3. Let M be a G - X -presentation of the space X . The category $\mathfrak{A}(X, G, M)$ is the category where the objects are equivariant G - X -modules for the presentation M , and the morphisms are norm limits of G -equivariant, bounded, finite propagation operators.

The category $\mathfrak{A}(X, G, M)$ and its ideal $\mathfrak{C}(X, G, M)$, are defined in an analogous way to the categories $\mathfrak{A}(X)$ and $\mathfrak{C}(X)$ in [2, p. 304] respectively. The ideal $\mathfrak{A}(X, G, M)$ is the C^* -category with the same objects as $\mathfrak{A}(X, G, M)$, and morphisms given by norm limits of G -equivariant, bounded, compactly supported operators.

For X compact, one can now proof the analogous of lemma 2.12 in [2]:

Lemma 4.4 (2.12 in [2]). *The C^* -algebra of endomorphisms of a non-trivial object in $\mathfrak{C}(X, G, M)$ is Morita equivalent to $C_r^*(G)$ and, therefore, their K -theories are isomorphic.*

Proof. Actually, the arguments for its proof in [7] are given for a proper cocompact action of G . \square

Let M be a simplicial complex, and let $G \times M \rightarrow M$ be a proper simplicial action of a discrete group G . Assume that the quotient M/G is compact. Let \mathfrak{F}_M the family generated by of (finite) subgroups of G having non empty fixed point set in M , i.e.

$$\mathfrak{F}_M = \{H < G \mid M^H \neq \emptyset\},$$

where

$$M^H = \{x \in M \mid hx = x, h \in H\}.$$

Definition 4.5. The action $G \times M \rightarrow M$ is said to be of bounded isotropy if the order of the elements in \mathfrak{F}_M is uniformly bounded, i.e. there is a constant c_M such that $|H| < c_M$ for every $H \in \mathfrak{F}_M$.

Lemma 4.6. *If the quotient $X = M/G$ is of bounded geometry and the action $G \times M \rightarrow M$ is of bounded isotropy, then M is of bounded geometry.*

Proof. Take a point $x \in M$ and let $S(x)$ the set of simplices containing x . Denote by $p : M \rightarrow M/G$ the projection on the quotient. Then $p(S(x)) = S(p(x))$ and, therefore, $\#S(x) \leq \#S(p(x)) \cdot c_M \leq N \cdot c_M$, where N is the bound on the number of simplices containing a point in M/G . \square

Remark 4.7. In [3], [4] it is shown that smooth manifolds with proper actions admit G -invariant triangulations. This means that one can regard a proper cocompact bounded geometry smooth G -manifold as an example of the above constructions. In this case one shall choose a triangulation such that every simplex is either fixed point-wise or permuted by the action.

Definition 4.8. Let X be a proper G -manifold and M a G -presentation of X . A Hilbert-Poincaré complex is equivariantly analytically controlled if it is analytically controlled over $(\mathfrak{A}(X, G, M), \mathfrak{C}(X, G, M))$, i.e. the modules in the complex are objects of these categories, the operator $B = b + b^*$ is controlled over $(\mathfrak{A}(X, G, M), \mathfrak{C}(X, G, M))$ and the duality operator S is a morphism in the category $\mathfrak{A}(X, G, M)$.

Theorem 4.9. *If the quotient $X = M/G$ is of bounded geometry and the action $G \times M \rightarrow M$ is of bounded isotropy, then its Higson-Roe non commutative signature is a homotopy and bordism invariant in the controlled category.*

Proof. As M is of bounded geometry, its simplicial chain and cochain complexes are geometrically controlled. The action either permutes or fixes simplices and is therefore unitary, and the fundamental cycle of such a triangulation is invariant. This means that the l^2 -chain complex of M is an example of an analytically controlled Poincaré complex.

In the case of bordism invariance, one shall assume that one has a triangulated bordism such that the simplices in the boundary coincide with the given triangulation of M .

The result now follows as a corollary of theorems 4.3 and 7.9 of [2]. \square

5. DIRECTED BORDISM INVARIANCE

In this section, we review the approach to bordism invariance of the coarse index due to C. Wulff [8] and extend it to the context of manifolds with proper actions of a discrete group.

Definition 5.1 (Directed bordism). Let N_1 and N_2 be proper, oriented G -manifolds of dimension n . Assume that they are furnished with a proper coarse structure.

A directed bordism from N_1 to N_2 is a proper G -manifold W , such that $\partial W = N_1 \amalg N_2$, the inclusions $i_1 : N_1 \rightarrow W$, $i_2 : N_2 \rightarrow W$ are coarse maps, and the coarse map i_2 is a coarse equivalence.

Definition 5.2. [Analytical c -Bordism groups] Let M be a proper G -space with an action of bounded isotropy. The analytical Bordism group $\Omega_n^{\text{an}, \text{eq}}(M)$ is the group with generators $(N, fE, b,)$, such that N is a proper, oriented manifold of bounded isotropy with a equivariant coarse map $f : N \rightarrow M$. E is G - X -Hilbert module with presentation N and $X = N/G$, and $b : E \rightarrow E$ is a controlled operator.

Two of such generators (N_1, f_1, E_1, b_1) and (N_2, f_2, E_2, b_2) are said to be c -bordant if there exists a directed bordism W from N_1 to N_2 , together with a coarse map $F : W \rightarrow M$, a map $E_i \rightarrow E$ covering the inclusions $N_i \rightarrow W$, and a controlled operator B restricting to f_i , respectively b_i .

If the space M is a proper oriented manifold of bounded isotropy, then it one may define the signature representative in the group $\Omega_n^{\text{an}, \text{eq}}(M)$ by taking $f = \text{id}$ and, for example, $E = \Omega_{L^2}^*(M)$, the L^2 -completion of the De Rham complex of M and b as the signature operator. One can also take $E' = C_*^{l^2}(M) \oplus C_*^{l^2}(M)$ and $b = B_S$ as in (3.2), where S is the Poincaré Duality homomorphism completion. Both choices coincide in terms of index by proposition 3.4.

Definition 5.3 (Analytical signature). The analytical signature is the class in $K^{n-1}(\mathfrak{A}(X, G, M)/\mathfrak{C}(X, G, M))$ of the Boundedly controlled operator associated to the Hilbert-Poincaré complex under the coarse assembly map.

In the following we shorten the notation $\mathfrak{A}(X, G, M)$, $\mathfrak{C}(X, G, M)$ by $\mathfrak{A}(M)$, $\mathfrak{C}(M)$ respectively.

We interpret now the main result of [8] in an equivariant setting:

Theorem 5.4. *The Analytical G -signature is Directed bordism invariant.*

Proof. The situation is completely analogous to [8], where it is seen to be a consequence of the naturality of the assembly map. Consider the diagram of G -equivariant inclusions, which are assumed to give coarse maps.

$$\begin{array}{ccc} N_1 & \longrightarrow & \partial W \longleftarrow N_2 \\ & & \downarrow \\ & & W \end{array}$$

The long exact sequence in K -theory of C^* algebras gives:

$$\begin{array}{ccccc} K_{p+1}(\mathfrak{A}(W/\partial W)/\mathfrak{C}(W/\partial W)) & \xrightarrow{\partial} & K_p(\mathfrak{A}(\partial W)/\mathfrak{C}(\partial W)) & \longrightarrow & K_p(\mathfrak{A}(W)/\mathfrak{C}(W)) \\ & & \downarrow A_{\partial W} & & \downarrow A_W \\ & & K_p(C^*(\partial W)) & \xrightarrow{i_*} & K_p(C^*(W)) \end{array}$$

Where the upper morphism ∂ is the connecting homomorphism, and the vertical morphisms are coarse assembly maps.

The functoriality of the Index Morphism= Assembly map gives thus that

$$A_W(i_1([b_1])) = A_W(i_2([b_2])).$$

□

6. MAPPING SURGERY TO ANALYSIS

In this section, we will state the main theorem of this note:

Theorem 6.1. *Let M be a proper G manifold with a bounded isotropy action. Assume that the quotient G/M has bounded geometry. Then, the following diagram is commutative*

$$\begin{array}{c} \Omega_n^{\text{an,eq}}(M) \xrightarrow{\quad \text{Alg Signature} \quad} K_n(C_*^r(G)) \\ \text{An Signature} \downarrow \quad \searrow \\ K_{n-1}(\mathfrak{A}_G(M)/\mathfrak{C}_G(M)) \xrightarrow{\omega_1} KK_G^n(C_0(M), \mathbb{C}) \xrightarrow{\omega_2} KK_G^n(C_0(\underline{E}G), \mathbb{C}) \xrightarrow{\mu} K_n(C_*^r(G)) \end{array}$$

where the map ω_1 is the isomorphism constructed in [7] (denoted by ω_4 in page 242), the group homomorphism ω_2 is induced by the up to G -equivariant homotopy unique map $M \rightarrow \underline{E}G$, and μ denotes the analytical Baum-Connes assembly map in KK -theory.

Proof. The analytical Assembly map $\mu : KK_G^n(C_0(\underline{E}G), \mathbb{C}) \rightarrow K_n(C_*^r(G))$ is given by the composite of the descent homomorphism $KK_G^n(C_0(\underline{E}G), \mathbb{C}) \rightarrow KK^n(C_0(\underline{E}G) \rtimes_r G, \mathbb{C} \rtimes_r G)$ followed by composing with the map given by the Kasparov product with the Mishchenko-Fomenko line bundle for $\underline{E}G$, $KK^n(C_0(\underline{E}G) \rtimes_r G, \mathbb{C} \rtimes_r G) \rightarrow KK^n(\mathbb{C}, C_*^r(G))$. By KK -theoretical homotopy invariance, the composite map

$$KK_G^n(C_0(M), \mathbb{C}) \xrightarrow{\omega_2} KK_G^n(C_0(\underline{E}G), \mathbb{C}) \xrightarrow{\mu} K_n(C_*^r(G))$$

agrees with the composite

$$KK_G^n(C_0(M), \mathbb{C}) \rightarrow KK^n(C_0(M) \rtimes_r G, \mathbb{C} \rtimes_r G) \rightarrow KK^n(\mathbb{C}, C_*^r(G)),$$

which consists of the descent homomorphism followed by the Kasparov product with a Mishchenko-Fomenko element for $C_0(M)$ (called w_5 and w_6 in [7], p. 242, respectively.) □

REFERENCES

- [1] Higson N., Roe J. *Mapping surgery to analysis I*. K-theory (2004) 33: 277–299, Springer 2005.
- [2] Higson N., Roe J. *Mapping surgery to analysis II*. K-theory (2004) 33: 301–324, Springer 2005.
- [3] S. Illman. *Existence and uniqueness of equivariant triangulations of smooth proper G -manifolds with some applications to equivariant Whitehead torsion*. J. Reine Angew. Math., 524:129–183, 2000.
- [4] T. Korppi. *Equivariant triangulations of differentiable and real-analytic manifolds with a properly discontinuous action*. Annales Academi acientiarum fennic mathematica dissertationes, number 141. Suomalainen Tiedeakatemia, Helsinki, 2005.
- [5] A.S. Mishchenko. *Signature of manifolds with proper action of a discrete group and the Hirzebruch type formula*. Conference: “Topology, Geometry, and Dynamics: Rokhlin Memorial”, Saint Petersburg, 2010.
- [6] Mishchenko A. S., *Homotopy invariants of non-simply connected manifolds. I. Rational invariants* Izv. Akad. Nauk SSSR Ser. Mat., 34 (1970), 501–514
- [7] J. Roe. Comparing Analytical Assembly maps. Quart. J. Math. 53 (2002), 241–248.
- [8] C. Wulff. Bordism Invariance of the Coarse index. Proc. American Mathematical Society. Vol. 140. number 8, 2012. Pages 2693–2697.

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